Chapter X

Waves and Metallic Boundary Conditions

A new technology has arisen in the last few decades that is based on the use of electromagnetic waves in the neighborhood of 10 cm wavelength. Many of the practical devices in use depend on the behavior of these waves when bounded by conducting surfaces. For example, the waves are generated in resonant cavities and directed to and from antennas in waveguides.

In this chapter we will discuss the basic principles of these devices and some of the easiest examples, but we will not develop the practical applications. The subjects emphasized are the boundary conditions imposed by good conductors; modes of radiation propagating between mirrors, propagating inside a waveguide of constant cross section, and standing in a resonant cavity; and the calculation of energy loss.

33. WAVEGUIDES AND RESONANT CAVITIES

Wave Propagation in Good Conductors

Preliminary to the discussion of metallic boundary conditions will be the consideration of how waves propagate inside conductors. The answer is "poorly," of course, but one needs to know the answer quantitatively in order to describe the reflection and loss that occur at the surface of a metal. Maxwell's equations, as they apply in the interior of a conductor with $\mathbf{J} = \sigma\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$, and $\mathbf{D} = \epsilon\mathbf{E}$, are

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \\ \frac{1}{\mu}\nabla \times \mathbf{B} &= \frac{4\pi}{c}\sigma\mathbf{E} + \frac{\epsilon}{c}\frac{\partial \mathbf{E}}{\partial t}, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \cdot \mathbf{D} &= 0. \end{aligned} \tag{33.1}$$

Traveling wave solutions, at a real frequency ω, can be found in the form

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \\ \mathbf{B} &= \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \end{aligned} \tag{33.2}$$

where k is in general complex, but $\hat{\mathbf{k}}$ is a real unit vector. All but the second of Maxwell's equations are satisfied if

$$\begin{aligned} \mathbf{k} \cdot \mathbf{E}_0 &= 0, \\ \mathbf{B}_0 &= \frac{c}{\omega}\mathbf{k} \times \mathbf{E}_0 . \end{aligned} \tag{33.3}$$

Then the dispersion results from using these fields in the second Maxwell equation:

$$\begin{aligned} \frac{ic}{\omega\mu}\mathbf{k} \times (\mathbf{k} \times \mathbf{E}_0) + i\omega\frac{\epsilon}{c}\mathbf{E}_0 &= \frac{4\pi}{c}\sigma\mathbf{E}_0 , \\ -i\frac{c}{\omega\mu}k^2 + i\omega\frac{\epsilon}{c} &= \frac{4\pi}{c}\sigma, \\ k^2 &= \epsilon\mu\frac{\omega^2}{c^2} + i4\pi\sigma\frac{\omega\mu}{c^2} . \end{aligned} \tag{33.4}$$

For a good conductor, by definition, $4\pi\sigma/\omega\epsilon$ is large compared to unity and

$$k^2 = i4\pi\sigma\frac{\omega\mu}{c^2} . \tag{33.5}$$

Neglecting $\epsilon\mu\omega^2/c^2$ in Eq. (33.4) is the same as neglecting the displacement current $c^{-1}\,\partial\mathbf{D}/\partial t$ in Maxwell's equations. A good conductor is therefore one in which the conduction current is larger than the displacement current; of course this depends on the frequency of the wave. Thus for a good conductor k is complex and given by

$$k = \frac{1+i}{\sqrt{2}} \sqrt{\frac{4\pi\sigma\omega\mu}{c^2}} . \tag{33.6}$$

The wave is attenuated with distance according to the factor

$$e^{ik\hat{\mathbf{k}}\cdot\mathbf{x}} = e^{i\sqrt{2\pi\sigma\omega\mu/c^2}\,\hat{\mathbf{k}}\cdot\mathbf{x}}\, e^{-\sqrt{2\pi\sigma\omega\mu/c^2}\,\hat{\mathbf{k}}\cdot\mathbf{x}} . \tag{33.7}$$

This means that the transmitted part of a wave impinging on the surface of a good conductor is damped exponentially as it propagates into the metal. The distance characteristic of the attenuation is

$$\delta = \frac{c}{\sqrt{2\pi\sigma\omega\mu}} , \tag{33.8}$$

which is known as the *skin depth*. The skin depth of a good conductor is small compared to a wavelength of the free radiation since

$$\begin{aligned} \frac{\delta}{\lambda} &= \frac{c}{\sqrt{2\pi\sigma\omega\mu}} \frac{\omega}{2\pi c} \\ &= \frac{1}{\pi} \sqrt{\frac{\omega\epsilon}{4\pi\sigma}} \sqrt{\frac{1}{2\epsilon\mu}} . \end{aligned} \tag{33.9}$$

The wave dies off to zero on a scale short compared to its wavelength outside the metal but on the same scale as its wavelength inside.

Zero-Order Surface Effects

In most problems it is both difficult and unnecessary to treat the phenomena that occur in the surface exactly. Instead, there is a general analysis of surface effects that can be made, for a good conductor, which leads to boundary conditions that determine the fields outside and leads to a recipe for calculating the losses due to penetration of the fields into the conductor. The surface effects will be studied in two stages of approximation.

The zero-order approximation is to suppose that there is an effective surface charge density σ and a surface current $\mathbf{K}$ (charge per second, per unit length transverse to the flow) on the metal. The boundary conditions are then obtained by integrating Maxwell's equations over small pill boxes or closed circuits straddling the boundary as shown in Fig. 33.1. For the normal component of **D**, one integrates over the

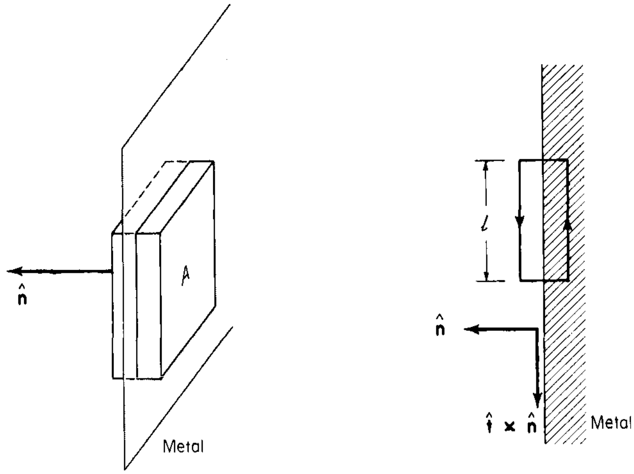

FIG. 33.1. *Geometry for obtaining boundary conditions from Maxwell's equations. Here* $\hat{\mathbf{n}}$ *is the unit normal out of the metal and* $\hat{\mathbf{t}}$ *is a unit tangential vector.*

pillbox using Gauss's theorem:

$$\begin{aligned} \nabla \cdot \mathbf{D} &= 4\pi\rho, \\ (\mathbf{D}_{\text{out}} - \mathbf{D}_{\text{in}}) \cdot \hat{\mathbf{n}} &= 4\pi\sigma, \\ \mathbf{D} \cdot \hat{\mathbf{n}} &= 4\pi\sigma \end{aligned} \tag{33.10}$$

in the limit as the dimensions of the pillbox shrink to zero. In the last line, **D** is the field just outside; the fields inside the conductor are being ignored in this approximation. Also, the pillbox thickness is supposed to be large compared to δ, and the effect of the fields inside is supposed to be summarized by σ and **K**. The same reasoning applied to the equation $\nabla \cdot \mathbf{B} = 0$ shows that

$$\begin{aligned} (\mathbf{B}_{\text{out}} - \mathbf{B}_{\text{in}}) \cdot \hat{\mathbf{n}} &= 0, \\ \mathbf{B} \cdot \hat{\mathbf{n}} &= 0, \end{aligned} \tag{33.11}$$

where **B** is the field just outside the conductor. For the tangential component of **H** one considers the equation

$$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \tag{33.12}$$

integrated over the area of a closed path that straddles the boundary as shown in Fig. 33.1. The size of the path parallel to the surface is l, and the size normal to the surface is negligible. If $\hat{\mathbf{t}}$ is the normal to the circuit in the right-hand sense, then application of Stokes's theorem shows that

$$\begin{aligned}
\int_{\ominus} d\mathbf{a} \cdot \nabla \times \mathbf{H} &= \frac{4\pi}{c}\int_{\ominus} d\mathbf{a} \cdot \mathbf{J}, \\
\oint \mathbf{H} \cdot dl &= \frac{4\pi}{c}\mathbf{K} \cdot \hat{\mathbf{t}}l, \\
(\mathbf{H}_{\text{out}} - \mathbf{H}_{\text{in}}) \cdot (\hat{\mathbf{t}} \times \hat{\mathbf{n}}) &= \frac{4\pi}{c}\mathbf{K} \cdot \hat{\mathbf{t}}, \\
[\hat{\mathbf{n}} \times (\mathbf{H}_{\text{out}} - \mathbf{H}_{\text{in}})] \cdot \hat{\mathbf{t}} &= \frac{4\pi}{c}\mathbf{K} \cdot \hat{\mathbf{t}}
\end{aligned} \tag{33.13}$$

applies in the limit of small l. The integral of $c^{-1}\,\partial\mathbf{D}/\partial t$ over the area of the circuit is proportional to the area of the circuit instead of to its length and does not contribute in this limit. Thus once again the displacement current is negligible in the effects due to a good conductor. Both of the vectors contracted with $\hat{\mathbf{t}}$ in Eqs. (33.13) lie in the surface, and since $\hat{\mathbf{t}}$ can be in any direction in the plane of the boundary, one must conclude that

$$\begin{aligned}
\hat{\mathbf{n}} \times (\mathbf{H}_{\text{out}} - \mathbf{H}_{\text{in}}) &= \frac{4\pi}{c}\mathbf{K}, \\
\hat{\mathbf{n}} \times \mathbf{H} &= \frac{4\pi}{c}\mathbf{K},
\end{aligned} \tag{33.14}$$

where in the last equation **H** is the field just outside the metal. Thus the tangential component of **H** gives the surface current per unit length. The last condition in the zero-order approximation comes from the equation

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{33.15}$$

and leads, by the same sort of argument, to

$$\hat{\mathbf{n}} \times \mathbf{E} = 0, \tag{33.16}$$

where **E** is the field just outside. In summary, the boundary conditions that one should use are that **E** is normal to the surface and **B** is tangential. These boundary conditions are sufficient to solve many waveguide problems. Once a problem is solved, the normal component of **D** and the tangential component of **H** determine the surface charge and current.

First-Order Surface Effects

The next approximation deals with the way that the fields make the change from their values outside to zero inside the conductor. This leads to an understanding of losses in waveguides and cavities; it turns out that the loss per unit area can be expressed in terms of $\mathbf{H}_{\parallel}$, the field just outside. Let $\mathbf{E}_c(\mathbf{x}, t)$ and $\mathbf{H}_c(\mathbf{x}, t)$ be the fields in the interior of the conductor. The time dependence of these fields is contained in the factor $e^{-i\omega t}$. For a good conductor, the displacement current in Maxwell's equations may be neglected and so

$$\begin{aligned} \nabla \times \mathbf{E}_c - i\frac{\omega\mu}{c}\mathbf{H}_c &= 0, \\ \nabla \times \mathbf{H}_c &= \frac{4\pi}{c}\sigma\mathbf{E}_c . \end{aligned} \tag{33.17}$$

The key to understanding the surface effects is to know that the fields inside the conductor vary more rapidly in the direction normal to the surface than in the directions parallel to the surface. The variation normal is on the scale of the skin depth whereas the variation parallel is on the scale of the wavelength outside the metal, since the boundary conditions have to be satisfied over the entire surface. The ratio is small for a good conductor, as shown by Eq. (33.9). Therefore, it makes sense to let

$$\nabla = -\hat{\mathbf{n}}\frac{\partial}{\partial \xi}, \tag{33.18}$$

when acting on the internal fields, where ξ is the coordinate along the normal into the metal. This holds in some degree of approximation measured by δ/λ and depending on details of the conductor geometry. In this case Eqs. (33.17) become

$$\mathbf{H}_c = \frac{ic}{\mu\omega}\hat{\mathbf{n}} \times \frac{\partial \mathbf{E}_c}{\partial \xi},$$
$$\mathbf{E}_c = -\frac{c}{4\pi\sigma}\hat{\mathbf{n}} \times \frac{\partial \mathbf{H}_c}{\partial \xi}. \tag{33.19}$$

According to the first equation $\mathbf{H}_c \cdot \hat{\mathbf{n}} = 0$, so that $\mathbf{H}_c$ is parallel to the boundary surface in this approximation. Eliminating $\mathbf{E}_c$ results in

$$\begin{aligned}\mathbf{H}_c &= \frac{ic}{\mu\omega}\hat{\mathbf{n}} \times \left(-\frac{c}{4\pi\sigma}\right)\left(\hat{\mathbf{n}} \times \frac{\partial^2 \mathbf{H}_c}{\partial \xi^2}\right) \\ &= \frac{ic^2}{4\pi\mu\omega\sigma}\frac{\partial^2 \mathbf{H}_c}{\partial \xi^2} \\ &= i\frac{\delta^2}{2}\frac{\partial^2 \mathbf{H}_c}{\partial \xi^2}.\end{aligned} \tag{33.20}$$

The solution of this equation that decreases into the material is given by

$$\mathbf{H}_c = \mathbf{H}_{\parallel}\, e^{-(1-i)\xi/\delta} \tag{33.21}$$

since $(-1 + i)^2$ is just $-i2$. We have taken the initial value of the solution to be $\mathbf{H}_{\parallel}$ because the internal fields are treated explicitly. Then there is no surface current so we have

$$\mathbf{n} \times (\mathbf{H}_{\text{out}} - \mathbf{H}_{\text{in}}) = 0, \tag{33.22}$$

and the value $\mathbf{H}_c$ at $\xi = 0$ becomes $\mathbf{H}_{\parallel}$, the tangential field just outside the conductor. The result is that, in this approximation, the fields in the metal are given by

$$\begin{aligned}\mathbf{H}_c &= \mathbf{H}_{\parallel}\, e^{-(1-i)\xi/\delta}, \\ \mathbf{E}_c &= -\frac{c}{4\pi\sigma}\hat{\mathbf{n}} \times \left(\frac{-1+i}{\delta}\right)\mathbf{H}_{\parallel}\, e^{-(1-i)\xi/\delta} \\ &= \sqrt{\frac{\mu\omega}{8\pi\sigma}}(1-i)\hat{\mathbf{n}} \times \mathbf{H}_{\parallel}\, e^{-(1-i)\xi/\delta}.\end{aligned} \tag{33.23}$$

It should be noted that the electric field inside the conductor is much smaller in magnitude than the magnetic field because ω/σ is small for a good conductor. This gives the details of the surface current since $\mathbf{J} = \sigma\mathbf{E}_c$. The losses in the walls of the conductor may be calculated by evaluating the Poynting vector just outside the metal. The power loss per unit area (the flux of free energy) is the time average of the component of the Poynting vector directed into the metal and is given by

$$\frac{dP}{da} = -\frac{c}{8\pi}\operatorname{Re}(\mathbf{E}_{\text{out}} \times \mathbf{H}_{\text{out}}^{*}) \cdot \hat{\mathbf{n}}. \tag{33.24}$$

Only the tangential components of the fields contribute in this expression. The tangential component of $\mathbf{E}$ is continuous so, in the present approximation,

$$\begin{aligned} \mathbf{E}_{\tan\,\text{out}} &= \mathbf{E}_{\tan\,\text{in}} \\ &= \mathbf{E}_c]_{\xi=0} \\ &= \sqrt{\frac{\mu\omega}{8\pi\sigma}}(1-i)\hat{\mathbf{n}} \times \mathbf{H}_{\parallel}\,. \end{aligned} \tag{33.25}$$

The tangential component of $\mathbf{H}$ just outside the surface is $\mathbf{H}_{\parallel}$ so the loss per unit area is

$$\begin{aligned} \frac{dP}{da} &= -\frac{c}{8\pi}\sqrt{\frac{\mu\omega}{8\pi\sigma}}\operatorname{Re}\{(1-i)[(\hat{\mathbf{n}} \times \mathbf{H}_{\parallel}) \times \mathbf{H}_{\parallel}^{*}] \cdot \hat{\mathbf{n}}\} \\ &= \frac{c}{8\pi}\sqrt{\frac{\mu\omega}{8\pi\sigma}}\operatorname{Re}\{(1-i)[\hat{\mathbf{n}}(\mathbf{H}_{\parallel} \cdot \mathbf{H}_{\parallel}^{*}) \cdot \hat{\mathbf{n}}]\} \\ &= \frac{\mu\omega\delta}{16\pi}\mathbf{H}_{\parallel} \cdot \mathbf{H}_{\parallel}^{*}. \end{aligned} \tag{33.26}$$

To solve a practical waveguide problem, one uses the boundary conditions $\mathbf{B}_{\text{norm}} = \mathbf{E}_{\tan} = 0$ to find the fields between the conducting surfaces. Then the losses can be calculated from Eq. (33.26) from the value of $\mathbf{H}_{\parallel}$. The energy that is lost goes into I^2R heating in the walls.

Propagation between Two Mirrors

Some of the physical properties of waveguides appear already in the simpler problem of propagation of light between two mirrors. Consider two plane mirrors located at $x = 0$ and $x = L$ with propagation in the XY-plane as shown in Fig. 33.2. The propagation takes place

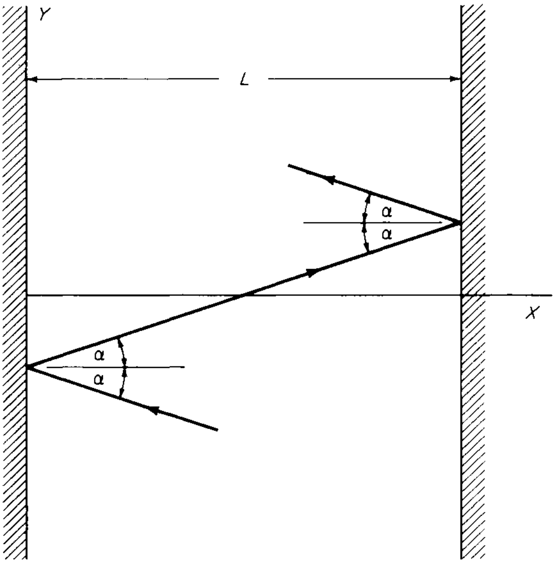

FIG. 33.2. *Propagation between two mirrors.*

mainly in the Y-direction and the light merely reflects back and forth in the X-direction. For light of wavelength $\lambda = 2\pi/k$ the figure shows that there are two wave vectors having components $(\pm k \cos \alpha, k \sin \alpha, 0)$ that should be considered. Here α is the angle of incidence or reflection at the mirrors. The space–time dependence of the fields associated with these waves are of the form $e^{i(\pm kx\cos\alpha + ky\sin\alpha - \omega t)}$ where ω is just ck. The two waves must superimpose in such a way that the boundary conditions

$$\begin{aligned} E_y(0, t) &= 0, \\ E_z(0, t) &= 0, \\ B_x(0, t) &= 0 \end{aligned} \tag{33.27}$$

are satisfied. Like boundary conditions apply at $x = L$. Waves that satisfy Eqs. (33.27) will be expressible as vector amplitudes times the space and time dependence

$$e^{i(kx\cos\alpha + ky\sin\alpha - \omega t)} - e^{i(-kx\cos\alpha + ky\sin\alpha - \omega t)} = i2 \sin(kx \cos \alpha)\, e^{i(ky\sin\alpha - \omega t)}. \tag{33.28}$$

The x-dependence here is that of a standing wave, and the solutions propagate (possibly with attenuation) in the Y-direction. The wave must also vanish at $x = L$ for any value of y so

$$kL \cos \alpha = n\pi, \tag{33.29}$$

where n is any positive integer. The different types of fields that result for different values of n are called different *modes* of the system. The angle of incidence and reflection α is determined by

$$\begin{aligned} c^2(k \cos \alpha)^2 + c^2(k \sin \alpha))^2 &= c^2k^2, \\ \left(\frac{n\pi c}{L}\right)^2 + (\omega \sin \alpha)^2 &= \omega^2. \end{aligned} \tag{33.30}$$

This equation permits α to be either real or purely imaginary. When α is real, $\sin \alpha$ is also real, and the wave propagates in the Y-direction. On the other hand if α is purely imaginary, so is $\sin \alpha$, and the wave is attenuated in the Y-direction. It is seen that α is real (or purely imaginary) when ω is greater (or less) than $n\pi c/L$. Thus the various modes have different cutoff frequencies as shown in Fig. 33.3.

The outstanding feature of propagation between mirrors is that there are modes, associated with the positive integers, and that each mode has a cutoff frequency below which energy cannot be transmitted. The

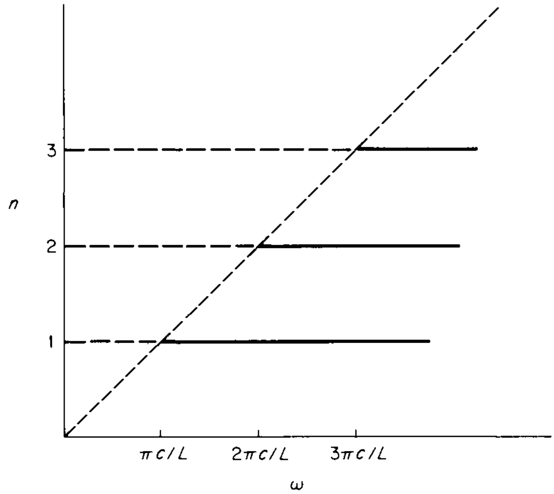

FIG. 33.3. *Modes of propagation between two mirrors.*

standing waves in the X-direction are observable, even at light frequencies. The observation can be made by inserting a photographic plate at a small angle to the mirrors. The maxima and minima of the fields along the perpendicular to the mirrors are then observably separated as fringes on the developed plate. These are known as *Lippmann fringes.* The dependence on y and t of the wave is contained in the exponential factor $e^{i(k'y-\omega t)}$, where k' is $k \sin \alpha$. This can be thought of as a wave solution of a one-dimensional equation. The phase velocity is then

$$\begin{aligned} v_p &= \frac{\omega}{k'} = \frac{\omega}{k \sin \alpha} \\ &= \frac{c}{\sin \alpha}. \end{aligned} \tag{33.31}$$

Thus the phase velocity at propagating frequencies is always greater than the speed of light and at cutoff it even becomes infinite. The dispersion equation in this equivalent one-dimensional problem is

$$\omega^2 = c^2 k'^2 + \left(\frac{n\pi c}{L}\right)^2. \tag{33.32}$$

The group velocity is obtained as usual by differentiating the dispersion equation

$$\begin{aligned} \omega\, d\omega &= c^2 k'\, dk', \\ \frac{\omega}{k'} \frac{d\omega}{dk'} &= c^2, \\ v_p v_g &= c^2. \end{aligned} \tag{33.33}$$

Thus the group velocity at propagating frequencies is always less than c, and at cutoff it vanishes. Energy ceases to be propagated in a mode at its cutoff frequency.

General Theory for Propagation in a Guide of Uniform Cross Section

A very convenient formulation can be developed for the general waveguide problem: Find solutions of Maxwell's equations in case the conductor containing the fields has translational symmetry along an axis. An example is sketched in Fig. 33.4.

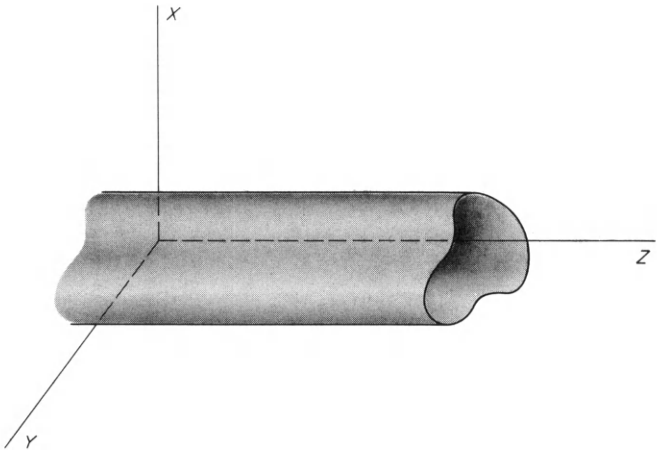

FIG. 33.4. *Waveguide with uniform cross section.*

Let the Z-axis be this symmetry axis so that the interior of the guide has the same cross section in any plane parallel to the XY-plane. Suppose that the interior of the guide is filled with a material which has uniform dielectric constant ϵ and permeability μ, although in most practical cases these are just unity. There are no charges or currents inside the waveguide so Maxwell's equations become

$$\begin{aligned} \nabla \times \mathbf{E} - i\frac{\omega}{c}\mathbf{B} &= 0, \\ \nabla \times \mathbf{B} + i\frac{\omega\mu\epsilon}{c}\mathbf{E} &= 0, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \cdot \mathbf{E} &= 0, \end{aligned} \tag{33.34}$$

for time dependence $e^{-i\omega t}$. Solutions are sought such that the tangential component of $\mathbf{E}$ and the normal component of $\mathbf{B}$ vanish at the walls of the guide.

Because of the translational symmetry of the problem, it is useful to set up a decomposition into components along and perpendicular to the Z-axis called the transverse and longitudinal parts, say

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_{\text{l}} + \mathbf{E}_{\text{tr}}, \\ \mathbf{B} &= \mathbf{B}_{\text{l}} + \mathbf{B}_{\text{tr}}, \\ \nabla &= \nabla_{\text{tr}} + \nabla_{\text{l}}. \end{aligned} \tag{32.35}$$

Here, the longitudinal part of the electric field, for example, is defined by

$$\mathbf{E}_l = (\hat{\mathbf{i}}_3 \cdot \mathbf{E})\hat{\mathbf{i}}_3 , \tag{33.36}$$

and its transverse part can be expressed as

$$\begin{aligned} \mathbf{E}_{tr} &= \mathbf{E} - \mathbf{E}_l \\ &= \mathbf{E} - \hat{\mathbf{i}}_3(\hat{\mathbf{i}}_3 \cdot \mathbf{E}) \\ &= -\hat{\mathbf{i}}_3 \times (\hat{\mathbf{i}}_3 \times \mathbf{E}). \end{aligned} \tag{33.37}$$

Similar formulas apply for $\mathbf{B}$ and ∇. For any two vectors or operators $\mathbf{v}$ and $\mathbf{w}$ decomposed this way one has

$$\begin{aligned} \mathbf{v}_l \cdot \mathbf{w}_{tr} &= 0, \\ \mathbf{v}_l \times \mathbf{w}_l &= 0. \end{aligned} \tag{33.38}$$

Furthermore $\mathbf{v}_{tr} \times \mathbf{w}_{tr}$ is longitudinal and $\mathbf{v}_{tr} \times \mathbf{w}_l$ is transverse. The cross product $\mathbf{v} \times \mathbf{w}$ can be decomposed as

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= (\mathbf{v}_l + \mathbf{v}_{tr}) \times (\mathbf{w}_l + \mathbf{w}_{tr}) \\ &= \mathbf{v}_l \times \mathbf{w}_{tr} + \mathbf{v}_{tr} \times \mathbf{w}_l + \mathbf{v}_{tr} \times \mathbf{w}_{tr} , \end{aligned}$$

so that

$$\begin{aligned} (\mathbf{v} \times \mathbf{w})_{tr} &= \mathbf{v}_l \times \mathbf{w}_{tr} + \mathbf{v}_{tr} \times \mathbf{w}_l , \\ (\mathbf{v} \times \mathbf{w})_l &= \mathbf{v}_{tr} \times \mathbf{w}_{tr} . \end{aligned} \tag{33.39}$$

Decomposing Maxwell's equations into transverse and longitudinal parts, one finds

$$\begin{aligned} \nabla_{tr} \times \mathbf{E}_{tr} - i\frac{\omega}{c}\mathbf{B}_l &= 0, \\ \nabla_{tr} \times \mathbf{B}_{tr} + i\frac{\omega\mu\epsilon}{c}\mathbf{E}_l &= 0 \end{aligned} \tag{33.40}$$

for the longitudinal parts of the curl equations and

$$\begin{aligned} \nabla_{tr} \times \mathbf{E}_l + \nabla_l \times \mathbf{E}_{tr} - i\frac{\omega}{c}\mathbf{B}_{tr} &= 0, \\ \nabla_{tr} \times \mathbf{B}_l + \nabla_l \times \mathbf{B}_{tr} + i\frac{\omega\mu\epsilon}{c}\mathbf{E}_{tr} &= 0 \end{aligned} \tag{33.41}$$

for their transverse parts. The divergence equations relate scalars and are simply

$$\nabla_{tr} \cdot \mathbf{B}_{tr} + \nabla_l \cdot \mathbf{B}_l = 0,$$
$$\nabla_{tr} \cdot \mathbf{E}_{tr} + \nabla_l \cdot \mathbf{E}_l = 0. \tag{33.42}$$

The problem is to solve these six equations.

The second of Eqs. (33.41) will be used to eliminate $\mathbf{E}_{tr}$ in the other equations. If the other field quantites are known it may be found from

$$\mathbf{E}_{tr} = \frac{ic}{\omega\mu\epsilon}(\nabla_{tr} \times \mathbf{B}_l + \nabla_l \times \mathbf{B}_{tr}). \tag{33.43}$$

When this is substituted into the first of Eqs. (33.40), and $\nabla_{tr} \cdot \mathbf{B}_{tr} = -\nabla_l \cdot \mathbf{B}_l$ is used, an equation in $\mathbf{B}_l$ alone results,

$$\begin{aligned} \mathbf{B}_l &= -i\frac{c}{\omega}\nabla_{tr} \times \mathbf{E}_{tr} \\ &= \frac{c^2}{\omega^2\mu\epsilon}[\nabla_{tr} \times (\nabla_{tr} \times \mathbf{B}_l) + \nabla_{tr} \times (\nabla_l \times \mathbf{B}_{tr})] \\ &= \frac{c^2}{\omega^2\mu\epsilon}[-\nabla_{tr}^2\mathbf{B}_l + \nabla_l(\nabla_{tr} \cdot \mathbf{B}_{tr})] \\ &= -\frac{c^2}{\omega^2\mu\epsilon}[\nabla_{tr}^2\mathbf{B}_l + \nabla_l(\nabla_l \cdot \mathbf{B}_l)]. \end{aligned} \tag{33.44}$$

When $\mathbf{E}_{tr}$ is eliminated from the transverse part of the first curl equation, the result is

$$\begin{aligned} \mathbf{B}_{tr} &= -i\frac{c}{\omega}(\nabla_{tr} \times \mathbf{E}_l + \nabla_l \times \mathbf{E}_{tr}) \\ &= -i\frac{c}{\omega}\nabla_{tr} \times \mathbf{E}_l + \frac{c^2}{\omega^2\mu\epsilon}[\nabla_l \times (\nabla_{tr} \times \mathbf{B}_l) + \nabla_l \times (\nabla_l \times \mathbf{B}_{tr})] \\ &= -i\frac{c}{\omega}\nabla_{tr} \times \mathbf{E}_l + \frac{c^2}{\omega^2\mu\epsilon}[\nabla_{tr}(\nabla_l \cdot \mathbf{B}_l) - \nabla_l^2\mathbf{B}_{tr}]. \end{aligned} \tag{33.45}$$

Thus far, the system of equations that must be solved has been reduced to

$$\mathbf{B}_l = -\frac{c^2}{\omega^2\mu\epsilon}[\nabla_{tr}^2\mathbf{B}_l + \nabla_l(\nabla_l \cdot \mathbf{B}_l)],$$
$$\nabla_{tr} \times \mathbf{B}_{tr} + i\frac{\omega\mu\epsilon}{c}\mathbf{E}_l = 0, \tag{33.46}$$

$$\mathbf{B}_{\mathrm{tr}} = -i\,\frac{c}{\omega}\,\boldsymbol{\nabla}_{\mathrm{tr}} \times \mathbf{E}_{\mathrm{l}} + \frac{c^2}{\omega^2\mu\epsilon}\,[\boldsymbol{\nabla}_{\mathrm{tr}}(\boldsymbol{\nabla}_{\mathrm{l}} \cdot \mathbf{B}_{\mathrm{l}}) - \nabla_{\mathrm{l}}^2\mathbf{B}_{\mathrm{tr}}],$$

$$\boldsymbol{\nabla}_{\mathrm{l}} \cdot \mathbf{B}_{\mathrm{l}} + \boldsymbol{\nabla}_{\mathrm{tr}} \cdot \mathbf{B}_{\mathrm{tr}} = 0,$$

$$\boldsymbol{\nabla}_{\mathrm{l}} \cdot \mathbf{E}_{\mathrm{l}} + \boldsymbol{\nabla}_{\mathrm{tr}} \cdot \mathbf{E}_{\mathrm{tr}} = 0,$$

where it is understood that $\mathbf{E}_{\mathrm{tr}}$ is given by Eq. (33.43).

The type of solution needed is one in which the fields have their z-dependence contained in the exponential factor $e^{\pm ikz}$, corresponding to propagation along the axis of the waveguide. Therefore, assuming the fields are of the form

$$\begin{aligned} \mathbf{E}(\mathbf{x}, t) &= \mathbf{E}_0(x, y)\, e^{i(\pm kz - \omega t)}, \\ \mathbf{B}(\mathbf{x}, t) &= \mathbf{B}_0(x, y)\, e^{i(\pm kz - \omega t)}, \end{aligned} \tag{33.47}$$

it follows that, for the longitudinal part of the gradient operator,

$$\begin{aligned} \boldsymbol{\nabla}_{\mathrm{l}} &= \hat{\mathbf{i}}_3(\hat{\mathbf{i}}_3 \cdot \boldsymbol{\nabla}) \\ &= \hat{\mathbf{i}}_3\,\frac{\partial}{\partial z} \\ &= \pm ik\hat{\mathbf{i}}_3\,, \\ \nabla_{\mathrm{l}}^2 &= -k^2. \end{aligned} \tag{33.48}$$

This brings the system of equations to be solved into the form

$$\begin{aligned} &\left(\nabla_{\mathrm{tr}}^2 - k^2 + \frac{\omega^2\mu\epsilon}{c^2}\right) B_z = 0, \\ &\boldsymbol{\nabla}_{\mathrm{tr}} \times \mathbf{B}_{\mathrm{tr}} + i\,\frac{\omega\mu\epsilon}{c}\,\mathbf{E}_{\mathrm{l}} = 0, \\ &\mathbf{B}_{\mathrm{tr}} = \frac{1}{\left(\dfrac{\mu\epsilon\omega^2}{c^2} - k^2\right)}\left(\boldsymbol{\nabla}_{\mathrm{tr}}\,\frac{\partial B_z}{\partial z} - i\,\frac{\mu\epsilon\omega}{c}\,\boldsymbol{\nabla}_{\mathrm{tr}} \times \hat{\mathbf{i}}_3 E_z\right), \end{aligned} \tag{33.49}$$

with $\mathbf{E}_{\mathrm{tr}}$ still given by Eq. (33.43). The divergencelessness of $\mathbf{B}$ is guaranteed by this set of equations since the first and third of them lead to

$$\begin{aligned} \boldsymbol{\nabla}_{\mathrm{tr}} \cdot \mathbf{B}_{\mathrm{tr}} + \boldsymbol{\nabla}_{\mathrm{l}} \cdot \mathbf{B}_{\mathrm{l}} &= \frac{1}{\dfrac{\mu\epsilon\omega^2}{c^2} - k^2}\,\nabla_{\mathrm{tr}}^2\,\frac{\partial B_z}{\partial z} + \boldsymbol{\nabla}_{\mathrm{l}} \cdot \mathbf{B}_{\mathrm{l}} \\ &= -\frac{\partial B_z}{\partial z} + \boldsymbol{\nabla}_{\mathrm{l}} \cdot \mathbf{B}_{\mathrm{l}} \\ &= 0. \end{aligned} \tag{33.50}$$

The divergence of **E** is also guaranteed to be zero as will be shown presently.

The transverse component of **E** can be rewritten, using the expression for $\mathbf{B}_{\text{tr}}$ of Eqs. (33.49) in Eq. (33.43), as

$$\mathbf{E}_{\text{tr}} = \frac{ic}{\omega\mu\epsilon} \nabla_{\text{tr}} \times \mathbf{B}_{\text{l}} + \frac{\dfrac{ic}{\omega\mu\epsilon}}{\left(\dfrac{\mu\epsilon\omega^2}{c^2} - k^2\right)} \times \left[\nabla_{\text{l}} \times \nabla_{\text{tr}} \frac{\partial B_z}{\partial z} - i\frac{\mu\epsilon\omega}{c} \nabla_{\text{l}} \times (\nabla_{\text{tr}} \times \mathbf{E}_{\text{l}})\right]. \tag{33.51}$$

Now since $\nabla_{\text{l}}\, \partial B_z/\partial z = \nabla_{\text{l}}(\nabla_{\text{l}} \cdot \mathbf{B}_{\text{l}}) = -k^2\mathbf{B}_{\text{l}}$, this becomes

$$\begin{aligned}\mathbf{E}_{\text{tr}} &= \frac{ic}{\omega\mu\epsilon} \nabla_{\text{tr}} \times \mathbf{B}_{\text{l}} + \frac{\dfrac{ic}{\omega\mu\epsilon}k^2}{\left(\dfrac{\mu\epsilon\omega^2}{c^2} - k^2\right)} \nabla_{\text{tr}} \times \mathbf{B}_{\text{l}} + \frac{1}{\left(\dfrac{\mu\epsilon\omega^2}{c^2} - k^2\right)} \nabla_{\text{tr}}(\nabla_{\text{l}} \cdot \mathbf{E}_{\text{l}}) \\ &= \frac{1}{\left(\dfrac{\mu\epsilon\omega^2}{c^2} - k^2\right)} \left(\nabla_{\text{tr}} \frac{\partial E_z}{\partial z} + i\frac{\omega}{c} \nabla_{\text{tr}} \times \hat{\mathbf{i}}_3 B_z\right).\end{aligned} \tag{33.52}$$

On the other hand, the second and third of Eqs. (33.49) lead to an equation in $\mathbf{E}_{\text{l}}$ alone:

$$\begin{aligned}\mathbf{E}_{\text{l}} &= i\frac{c}{\omega\mu\epsilon} \nabla_{\text{tr}} \times \mathbf{B}_{\text{tr}} \\ &= i\frac{c}{\omega\mu\epsilon} \frac{1}{\left(\dfrac{\mu\epsilon\omega^2}{c^2} - k^2\right)} \left[-i\frac{\mu\epsilon\omega}{c} \nabla_{\text{tr}} \times (\nabla_{\text{tr}} \times \mathbf{E}_{\text{l}})\right],\end{aligned}$$

$$\begin{aligned}\left(\frac{\mu\epsilon\omega^2}{c^2} - k^2\right) \mathbf{E}_{\text{l}} &= -\nabla_{\text{tr}}^2 \mathbf{E}_{\text{l}}, \\ \left(\nabla_{\text{tr}}^2 - k^2 + \frac{\omega^2\mu\epsilon}{c^2}\right) \mathbf{E}_{\text{l}} &= 0.\end{aligned} \tag{33.53}$$

This, together with Eq. (33.52), implies that the divergence of **E** vanishes; the proof is the same as that for the divergence of **B**. Thus the full content of Maxwell's equations, provided that the z-dependence of the solutions is contained in the factor $e^{\pm ikz}$, is contained in the following system of equations:

$$\mathbf{E}_{\text{tr}} = \frac{1}{\mu\epsilon \dfrac{\omega^2}{c^2} - k^2}\left(\nabla_{\text{tr}} \frac{\partial E_z}{\partial z} + i\frac{\omega}{c}\nabla_{\text{tr}} \times \hat{\mathbf{i}}_3 B_z\right),$$

$$\mathbf{B}_{\text{tr}} = \frac{1}{\mu\epsilon \dfrac{\omega^2}{c^2} - k^2}\left(\nabla_{\text{tr}} \frac{\partial B_z}{\partial z} - i\mu\epsilon\frac{\omega}{c}\nabla_{\text{tr}} \times \hat{\mathbf{i}}_3 E_z\right),$$

$$\left(\nabla_{\text{tr}}^2 - k^2 + \mu\epsilon\frac{\omega^2}{c^2}\right) E_z = 0, \tag{33.54}$$

$$\left(\nabla_{\text{tr}}^2 - k^2 + \mu\epsilon\frac{\omega^2}{c^2}\right) B_z = 0.$$

The boundary conditions that must be imposed on solutions of this system are that the tangential component of **E** and the normal component of **B** must vanish on the walls of the guide. The axis of the waveguide is the Z-direction so one of the components of $\mathbf{E}_{\text{tan}}$ is just E_z. Hence

$$E_z = 0 \qquad \text{on the boundary,} \tag{33.55}$$

partly specifies the problem. It will be shown later that this is all that is needed for the electric field because $\mathbf{E}_{\text{tan}} = \mathbf{i}_3 E_z$ holds on the surface as long as the boundary condition on **B** is satisfied. The normal component of **B** is contained in

$$\mathbf{B}_{\text{tr}} = \frac{1}{\left(\mu\epsilon \dfrac{\omega^2}{c^2} - k^2\right)}\left(\pm ik\nabla_{\text{tr}} B_z + i\mu\epsilon\frac{\omega}{c}\mathbf{i}_3 \times \nabla_{\text{tr}} E_z\right), \tag{33.56}$$

where $\partial B_z/\partial z$ has been written as $\pm ikB_z$. Since E_z vanishes on the surface, $\nabla_{\text{tr}} E_z$ is normal there and hence $\hat{\mathbf{i}}_3 \times \nabla_{\text{tr}} E_z$ is tangential at the boundary. Thus the second term in Eq. (33.56) does not contribute to the normal component of **B** at the wall and so the boundary condition that must be satisfied by **B** is

$$\begin{aligned}\mathbf{n}\cdot\mathbf{B}_{\text{tr}} &= \frac{1}{\left(\mu\epsilon \dfrac{\omega^2}{c^2} - k\right)}(\pm ik)(\mathbf{n}\cdot\nabla_{\text{tr}})B_z \\ &= 0 \qquad \text{on the boundary.}\end{aligned} \tag{33.57}$$

That is, $\partial B_z/\partial n$ must vanish on the walls of the waveguide. There are two boundary value problems to be solved here. It is interesting that the one for E_z is of the Dirichlet type while that for B_z has the Neumann form. In principle the specification of the boundary conditions is yet incomplete, for $\mathbf{E}_{\mathrm{tr}}$ might conceivably have a component tangential to the wall. To see that this is not so, one looks at Eqs. (33.54) for $\mathbf{E}_{\mathrm{tr}}$ and first notes that

$$\nabla_{\mathrm{tr}} \frac{\partial E_z}{\partial z} = \pm ik\nabla_{\mathrm{tr}} E_z \tag{33.58}$$

is normal at the surface because E_z vanishes over the surface. The other term in the expression for $\mathbf{E}_{\mathrm{tr}}$ in terms of E_z and B_z in Eqs. (33.54) does not contribute to $\mathbf{E}_{\mathrm{tan}}$ either because

$$\begin{aligned} \mathbf{n} \times (\nabla_{\mathrm{tr}} \times \mathbf{B}_{\mathrm{l}}) &= \nabla_{\mathrm{tr}}(\mathbf{n} \cdot \mathbf{B}_{\mathrm{l}}) - (\mathbf{n} \cdot \nabla_{\mathrm{tr}})\mathbf{B}_{\mathrm{l}} \\ &= -\hat{\mathbf{i}}_3 \frac{\partial B_z}{\partial n} \\ &= 0 \qquad \text{on the boundary.} \end{aligned} \tag{33.59}$$

Thus $\nabla_{\mathrm{tr}} \times \mathbf{B}_{\mathrm{l}}$ is normal at the surface and it follows that $\mathbf{E}_{\mathrm{tr}}$ is also normal there. The boundary conditions are therefore completely satisfied when E_z and $\partial B_z/\partial n$ vanish at the surface.

Since the system is linear, one can pick either E_z or B_z to be identically zero and find solutions of the remaining nontrivial boundary value problem. The general problem is then solved by a superposition of these two types of solution. The two different types usually have different dispersions anyway and so it is better to solve them separately. The solutions found when B_z is set equal to zero are called *TM modes* since the magnetic field is then purely transverse. The Dirichlet boundary value problem,

$$\begin{aligned} \left(\nabla_{\mathrm{tr}}^2 + \mu\epsilon \frac{\omega^2}{c^2} - k^2\right) E_z &= 0, \\ E_z &= 0 \qquad \text{on the boundary,} \end{aligned} \tag{33.60}$$

must be solved for the TM modes for a given geometry of the waveguide. There will be solutions only for special values of $(\mu\epsilon\omega^2/c^2) - k^2$. These eigenvalues depend on two numbers, since the boundary value problem is two-dimensional, and so the dispersion of the TM modes is of the form

$$\mu\epsilon \frac{\omega^2}{c^2} - k^2 = \gamma_{mn}^2 , \tag{33.61}$$

where m and n are the serial numbers and γ_{mn}^2 are the characteristic values. The cutoff frequencies of these modes are given by

$$\omega_{mn} = \frac{c}{\sqrt{\mu\epsilon}} \gamma_{mn} . \tag{33.62}$$

Once E_z is determined as an eigenfunction corresponding to the eigenvalue γ_{mn}^2 , the transverse fields can be constructed from

$$\begin{aligned} \mathbf{E}_{\text{tr}} &= \frac{1}{\gamma_{mn}^2} \nabla_{\text{tr}} \frac{\partial E_z}{\partial z} \\ &= \pm \frac{ik}{\gamma_{mn}^2} \nabla_{\text{tr}} E_z , \\ \mathbf{B}_{\text{tr}} &= \frac{1}{\gamma_{mn}^2} i\mu\epsilon \frac{\omega}{c} \hat{\mathbf{i}}_3 \times \nabla_{\text{tr}} E_z \\ &= \pm \mu\epsilon \frac{\omega}{ck} \hat{\mathbf{i}}_3 \times \mathbf{E}_{\text{tr}} . \end{aligned} \tag{33.63}$$

On the other hand, when one takes E_z to be zero the electric field is purely transverse and the propagation is said to be in *TE modes.* The Neumann boundary value problem,

$$\begin{gathered} \left(\nabla_{\text{tr}}^2 + \mu\epsilon \frac{\omega^2}{c^2} - k^2\right) B_z = 0, \\ \frac{\partial B_z}{\partial n} = 0 \qquad \text{on the boundary,} \end{gathered} \tag{33.64}$$

specifies the TE modes for a particular waveguide geometry. Again there will be eigenvalues of the problem giving the dispersion for these modes

$$\mu\epsilon \frac{\omega^2}{c^2} - k^2 = \gamma_{mn}^2 . \tag{33.65}$$

When the eigenfunction solution for B_z corresponding to the eigenvalue γ_{mn} is determined, the transverse fields can be found easily from

$$\begin{aligned} \mathbf{B}_{\rm tr} &= \frac{1}{\gamma_{mn}^2} \nabla_{\rm tr} \frac{\partial B_z}{\partial z} \\ &= \pm \frac{ik}{\gamma_{mn}^2} \nabla_{\rm tr} B_z , \\ \mathbf{E}_{\rm tr} &= \frac{1}{\gamma_{mn}^2} \left(-i \frac{\omega}{c} \hat{\mathbf{i}}_3 \times \nabla_{\rm tr} B_z \right) \\ &= \mp \frac{\omega}{ck} \hat{\mathbf{i}}_3 \times \mathbf{B}_{\rm tr} . \end{aligned} \tag{33.66}$$

This completes the formulation of the theory for propagation in a waveguide of uniform cross section. It has been shown that there are two types of solutions and the boundary value problems for determining these TM and TE modes have been set up.

Modes in a Rectangular Guide

The rectangular guide is a mathematically convenient and practically relevant special case of the waveguide with uniform cross section. Suppose that the walls of the waveguide are conducting planes located at $x = 0$, $x = a$ and at $y = 0$, $y = b$ as shown in Fig. 33.5.

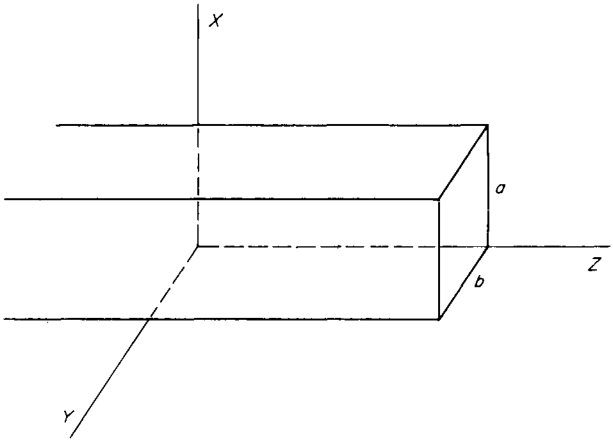

FIG. 33.5. *The rectangular waveguide.*

The TM modes are determined by the Dirichlet problem

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right) E_z = 0, \qquad E_z = 0 \quad \text{on the boundary.} \tag{33.67}$$

This boundary value problem is easily solved by the method of separation of variables. One first separates the x- and y-dependence in the differential equation by assuming a solution of the form

$$E_z(\mathbf{x}, t) = X(x)\, Y(y)\, e^{\pm ikz - i\omega t}. \tag{33.68}$$

The equation becomes

$$\frac{Y\, d^2X}{dx^2} + X \frac{d^2Y}{dy^2} + \gamma^2 XY = 0, \qquad \frac{X''}{X} + \frac{Y''}{Y} + \gamma^2 = 0. \tag{33.69}$$

Since X''/X is a function of x alone and Y''/Y is a function of y alone, each must equal a constant if the sum of them is constant. Thus X''/X is some number, say $-\alpha^2$, and the solution must be

$$X(x) = A \sin \alpha x. \tag{33.70}$$

Since E_z must vanish at $x = 0$, the cosine solution of the differential equation has been discarded at this point. Furthermore E_z must also vanish at $x = a$ so

$$\alpha a = m\pi; \qquad m = 1, 2, 3, \ldots . \tag{33.71}$$

With α thus established, the differential equation for Y and its solution are determined to be

$$Y'' + (\gamma^2 - \alpha^2) Y = 0, \qquad Y(y) = A' \sin \sqrt{\gamma^2 - \alpha^2}\, y. \tag{33.72}$$

The solution that vanishes at $y = 0$ has been selected here. It must also vanish at $y = b$ so

$$\sqrt{\gamma^2 - \alpha^2}\, b = n\pi; \qquad n = 1, 2, 3, \ldots . \tag{33.73}$$

This determines the eigenvalue γ_{mn} as a function of the two integers m and n. Thus the solution of the eigenvalue problem, with the product of constants AA' replaced by a single constant E_0, is

$$E_z = E_0 \sin m\pi \frac{x}{a} \sin n\pi \frac{y}{b} e^{\pm ikz - i\omega t},$$
$$\gamma_{mn}^2 = \left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{a}\right)^2; \qquad m, n = 1, 2, 3, \ldots . \tag{33.74}$$

Here m and n range independently over the positive integers, zero not permitted, and each pair specifies a possible TM mode in the waveguide. The transverse components of the fields in the TM modes of the rectangular guide are

$$E_x = \pm \frac{ik}{\gamma_{mn}^2} \frac{\partial E_z}{\partial x}$$
$$= \pm \frac{ik}{\gamma_{mn}^2} \frac{m\pi}{a} E_0 \cos m\pi \frac{x}{a} \sin n\pi \frac{y}{b} e^{\pm ikz - i\omega t},$$
$$E_y = \pm \frac{ik}{\gamma_{mn}^2} \frac{n\pi}{b} E_0 \sin m\pi \frac{x}{a} \cos n\pi \frac{y}{b} e^{\pm ikz - i\omega t}, \tag{33.75}$$
$$B_x = \mp \mu\epsilon \frac{\omega}{ck} E_y,$$
$$B_y = \pm \mu\epsilon \frac{\omega}{ck} E_x .$$

The TE modes in the rectangular guide are found by setting E_z equal to zero and solving the eigenvalue problem

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right) B_z = 0,$$
$$\frac{\partial B_z}{\partial n} = 0 \qquad \text{on the boundary.} \tag{33.76}$$

This problem is also easily solved by assuming a solution of the form

$$B_z(\mathbf{x}, t) = X(x)\, Y(y)\, e^{\pm ikz - i\omega t}, \tag{33.77}$$

so that the differential equation becomes

$$\frac{X''}{X} + \frac{Y''}{Y} + \gamma^2 = 0. \tag{33.78}$$

Now X''/X must be a constant, say $-\alpha^2$, and the solution of the equation

$$X'' + \alpha^2 X = 0 \tag{33.79}$$

has to be taken to be

$$X(x) = A \cos \alpha\, x, \tag{33.80}$$

since the boundary conditions require $\partial B_z/\partial x$ to vanish when x is zero. To obtain $\partial B_z/\partial x$ to vanish at $x = a$ as well, one must have

$$\alpha a = m\pi; \qquad m = 0, 1, 2, \ldots . \tag{33.81}$$

There is one qualitative difference between TM and TE modes: The integers specifying the number of half wavelengths of the radiation in the guide in the X- and Y-directions may be zero in the latter case. The $m = 0$ possibility leads, in Eq. (33.80), to a constant for $X(x)$ and is acceptable for the Neumann boundary conditions. The same development applies to $Y(y)$ and so the eigenvalue problem is solved by

$$\begin{aligned} B_z &= B_0 \cos m\pi\frac{x}{a} \cos n\pi\frac{y}{b}\, e^{\pm ikz - i\omega t}, \\ \gamma_{mn}^2 &= \left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{a}\right)^2; \qquad m, n = 0, 1\ 2, \ldots . \end{aligned} \tag{33.82}$$

However one must reconsider whether both m and n being zero at the same time is an acceptable solution because this leads to $\gamma = 0$. In the general development the equations were divided by γ_{mn} in several places, so the solution will not necessarily apply. One must reconsider the problem almost from the beginning with $(\mu\epsilon\omega^2/c^2) - k^2 = 0$. The derivation will not be given here, but the result is that there is no solution with $(\mu\epsilon\omega^2/c^2) - k^2 = 0$ in a waveguide problem if the cross section of the guide is simply connected. If the guide is not simply connected (for example, a coaxial cable is not), then there is a TEM mode. For this mode, both the electric and magnetic fields are transverse, and the TEM mode has no cutoff. It is the only propagating mode at low frequency in such a guide. Indeed a guide which is not simply connected can be thought of as possessing a return path for waves at zero frequency, that is, direct current.

In the rectangular guide the possible modes are TE_{m0}, TE_{0n}, TE_{mn}, and TM_{mn}, where m and n range over the positive integers. The dispersion rule is

$$\mu\epsilon\frac{\omega^2}{c^2} - k^2 = \gamma_{mn}^2$$
$$= \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2.$$

When a is greater than b, the lowest cutoff frequency is in the TE_{10} mode with $\gamma = \pi/a$. Since the TE_{10} mode enters into a subsequent discussion of the lowest resonance of a rectangular cavity, explicit formulas for the fields in this mode will be given. For this mode one has

$$\frac{\pi^2}{a^2} = \mu\epsilon\frac{\omega^2}{c^2} - k^2, \tag{33.83}$$

and the fields are, from Eqs. (33.82) and (33.66),

$$\begin{aligned} B_z &= B_0 \cos \pi\frac{x}{a}\, e^{\pm ikz - i\omega t}, \\ B_x &= \mp\frac{ika}{\pi} B_0 \sin \pi\frac{x}{a}\, e^{\pm ikz - i\omega t}, \\ B_y &= 0, \\ E_x &= \pm\frac{\omega}{ck} B_y \\ &= 0, \\ E_y &= \mp\frac{\omega}{ck} B_x \\ &= \frac{i\omega a}{\pi c} B_0 \sin \pi\frac{x}{a}\, e^{\pm ikz - i\omega t}. \end{aligned} \tag{33.84}$$

The Rectangular Resonant Cavity with Only TE_{10} above Cutoff

At sufficiently low frequency only the TE_{10} mode propagates in the rectangular waveguide. Consider this case so there will be only one mode to keep in mind. One can consider walling up the waveguide with conducting planes at $z = 0$ and $z = d$ to make a resonant cavity as shown in Fig. 33.6. In this situation, the traveling wave solutions with z-dependence $e^{\pm ikz}$ can be combined so as to make B_z and E_y vanish at the ends of the cavity. This satisfies all necessary boundary conditions

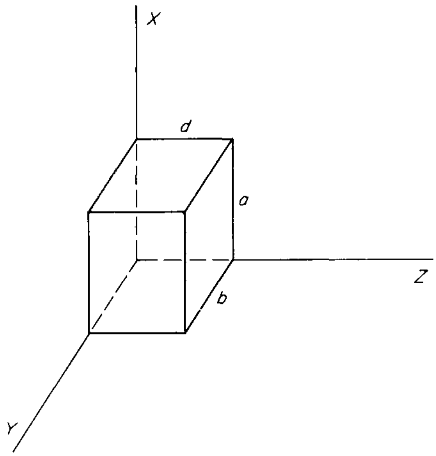

FIG. 33.6. *The rectangular resonant cavity.*

since E_x is zero in the TE_{10} mode of the rectangular guide. In fact the difference of the two oppositely propagating solutions divided by $i2$ is the required form and the nonvanishing fields are

$$\begin{aligned} B_z &= B_0 \cos \pi \frac{x}{a} \sin kz\, e^{-i\omega t}, \\ B_x &= -\frac{ka}{\pi} B_0 \sin \pi \frac{x}{a} \cos kz\, e^{-i\omega t}, \\ E_y &= i \frac{\omega a}{\pi c} B_0 \sin \pi \frac{x}{a} \sin kz\, e^{-i\omega t}. \end{aligned} \tag{33.85}$$

To allow B_z and E_y to vanish at $z = 0$ and $z = d$, one must have

$$kd = \pi p; \qquad p = 1, 2, 3, \ldots . \tag{33.86}$$

The case $p = 0$ gives the trivial solution with all fields vanishing everywhere. For $p = 1$ in Eq. (33.86), one has $k = \pi/d$ and the vibration can occur only at the special frequency given by

$$\begin{aligned} \frac{\pi^2}{a^2} &= \mu\epsilon \frac{\omega^2}{c^2} - \frac{\pi^2}{d^2}, \\ \omega &= \frac{c\pi}{\sqrt{\mu\epsilon}} \sqrt{\frac{1}{a^2} + \frac{1}{d^2}}. \end{aligned} \tag{33.87}$$

This is the lowest resonance of the rectangular cavity. The nonzero fields in this case are

$$\begin{aligned} B_z &= B_0 \cos \pi \frac{x}{a} \sin \pi \frac{z}{d} e^{-i\omega t}, \\ B_x &= -\frac{a}{d} B_0 \sin \pi \frac{x}{a} \cos \pi \frac{z}{d} e^{-i\omega t}, \\ E_y &= i \frac{\omega a}{\pi c} B_0 \sin \pi \frac{x}{a} \sin \pi \frac{z}{d} e^{-i\omega t}. \end{aligned} \tag{33.88}$$

This is just a special case; there are many resonant modes for various values of m, n, and p.

The losses that occur in the walls of the cavity may be discussed in terms of the quality factor Q which is defined by

$$\begin{aligned} Q &= 2\pi \left(\frac{\text{energy stored}}{\text{energy lost per cycle}} \right) \\ &= \omega_0 \left(\frac{\text{energy stored}}{\text{average power loss}} \right). \end{aligned} \tag{33.89}$$

The energy referred to here is the free energy of the fields in the cavity, and ω_0 is the natural frequency of the system. We will consider the typical case in which the losses in a cycle are small compared to the stored energy and in which Q has a numerical value which is independent of the amplitude of the vibration. The Q of the resonant cavity will be calculated for the lowest resonant frequency as an example but first there are some general remarks to be made: Let U be the energy present in the field. Since there are losses, this decreases slowly and must be a function of time $U(t)$. There is a range of frequencies around ω_0 introduced into the system by the losses. Since the quality factor is given in terms of $U(t)$ and the power loss, which is just $-dU(t)/dt$, one has

$$\begin{aligned} Q &= \omega_0 \frac{U}{\left(-\frac{dU}{dt}\right)}, \\ \frac{dU}{dt} &= -\frac{\omega_0}{Q} U, \\ U &= U_0 e^{-\frac{\omega_0}{Q} t}. \end{aligned} \tag{33.90}$$

Thus Q/ω_0 is the time constant for the excitation in the cavity. The fields inside do not have a pure sinusoidal time-dependence because of the exponential damping. Any one of the fields at a point inside the cavity has a time-dependence something like

$$E(t) = \begin{cases} 0, & t < 0, \\ E_0 e^{-\frac{\omega_0}{2Q}t} \cos \omega_0 t, & t > 0 \end{cases} \tag{33.91}$$

in the situation where the mode is excited in the cavity at $t = 0$ and then allowed to decay freely. One can make a Fourier analysis of this field to see the range of frequencies which are present in it (this calculation was done in solving Problem 5, Section 31); the result is shown in Fig. 33.7. The full width at half-maximum in the square of the amplitude of the Fourier transform is $\Delta\omega = \omega_0/Q$.

One needs to make use of the explicit fields of a particular mode to calculate the quality factor. For the standing TE_{10} mode with fields given by Eqs. (33.88), the stored energy in the cavity is

$$\begin{aligned} U &= \frac{1}{16\pi} \int_{\circledast} d^3x \left[\epsilon \mathbf{E} \cdot \mathbf{E}^* + \frac{1}{\mu} \mathbf{B} \cdot \mathbf{B}^*\right] \\ &= \frac{1}{16\pi} \int_{\circledast} d^3x \left[\epsilon \left(\frac{\omega a}{\pi c}\right)^2 B_0^2 \sin^2 \pi \frac{x}{a} \sin^2 \pi \frac{z}{d} + \frac{1}{\mu} B_0^2 \cos^2 \pi \frac{x}{a} \sin^2 \pi \frac{z}{d} \right. \\ &\quad \left. + \frac{1}{\mu} B_0^2 \left(\frac{a}{d}\right)^2 \sin^2 \pi \frac{x}{a} \cos^2 \pi \frac{z}{d}\right] \\ &= \frac{1}{64\pi} B_0^2 abd \left[\epsilon \left(\frac{\omega a}{\pi c}\right)^2 + \frac{1}{\mu} + \frac{1}{\mu} \left(\frac{a}{d}\right)^2\right] \\ &= \frac{1}{32\pi} B_0^2 \frac{abd}{\mu} \left(1 + \frac{a^2}{d^2}\right). \end{aligned} \tag{33.92}$$

The spatial integrals are easily performed. For example, the average value of $\sin^2 \pi x/a$ is $\frac{1}{2}$ and the range of its integration is from 0 to a so this factor contributes $\frac{1}{2}a$. The resonant frequency, where this calculation applies, satisfies $(a\omega/c\pi)^2 = (1 + a^2/d^2)/\mu\epsilon$. To obtain the average power loss, one supposes that the metal has permeability μ_c and applies Eq. (33.26)

$$\frac{dP}{da} = \frac{\mu_c \omega_0 \delta}{16\pi} \mathbf{H}_{\parallel} \cdot \mathbf{H}_{\parallel}^*, \tag{33.93}$$

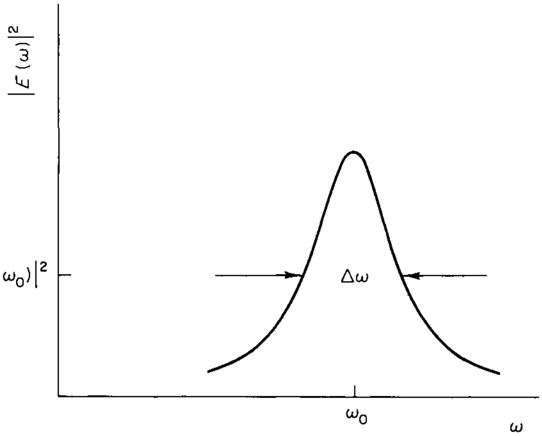

FIG. 33.7. *Amplitude of the Fourier transform of an exponentially damped sinusoidal wave.*

where δ, the skin depth in the metal, is $c/\sqrt{2\pi\sigma\omega_0\mu_c}$. The amounts absorbed by opposite faces in the cavity are equal so

$$
\begin{aligned}
P &= \frac{\mu_c\omega_0\delta}{16\pi\mu^2}\int da\,\mathbf{B}_\parallel\cdot\mathbf{B}_\parallel^* \\
&= \frac{\mu_c\omega_0\delta}{16\pi\mu^2}\left\{2\int_0^a dx\int_0^b dy\,|B_x|^2\right]_{z=0} + 2\int_0^a dx\int_0^d dz\,(|B_x|^2+|B_z|^2)\Big]_{y=0} \\
&\quad + 2\int_0^b dy\int_0^d dz\,|B_z|^2\Big]_{x=0}\Big\}, \\
&= \frac{\mu_c\omega_0\delta}{16\pi\mu^2}\left[\left(\frac{a}{d}\right)^2 ab + \left(\frac{a}{d}\right)^2\frac{ad}{2} + \frac{ad}{2} + bd\right]B_0^2 \\
&= \frac{\mu_c\omega_0\delta}{16\pi\mu^2}\left[a\left(b+\frac{d}{2}\right)\left(\frac{a}{d}\right)^2 + d\left(b+\frac{a}{2}\right)\right]B_0^2.
\end{aligned}
\tag{33.94}
$$

Thus the quality factor $Q = \omega_0 U/P$ is

$$
\begin{aligned}
Q &= \frac{\omega_0}{32\pi}B_0^2\frac{abd}{\mu}\left(1+\frac{a^2}{d^2}\right)\frac{16\pi\mu^2}{\mu_c\omega_0\delta B_0^2}\frac{1}{\left[a\left(b+\frac{d}{2}\right)\left(\frac{a}{d}\right)^2 + d\left(b+\frac{a}{2}\right)\right]} \\
&= \frac{\mu}{\mu_c}\frac{abd}{2\delta}\frac{\left(1+\frac{a^2}{d^2}\right)}{\left[a\left(b+\frac{d}{2}\right)\left(\frac{a}{d}\right)^2 + d\left(b+\frac{a}{2}\right)\right]}
\end{aligned}
\tag{33.95}
$$

for this particular mode of the rectangular cavity. There is a conventional way of writing the quality factor for a cavity. Since the area of the walls is $A = 2(ab + bd + ad)$ and the volume is $V = abd$, if one defines a dimensionless function of a, b, and d by

$$G = \left(1 + \frac{a^2}{d^2}\right) \frac{(ab + bd + ad)}{\left[a\left(b + \frac{d}{2}\right)\left(\frac{a}{d}\right)^2 + d\left(b + \frac{a}{2}\right)\right]}, \tag{33.96}$$

then the Q of the cavity becomes

$$Q = \frac{\mu}{\mu_c} \frac{V}{A\delta} G. \tag{33.97}$$

This is appropriate because G specifies the dependence of Q on the shape of the cavity. Otherwise the quality factor depends on the ratio of permeabilities μ/μ_c and on $V/A\delta$, which is essentially the ratio of the volume in which the field energy is conserved to that in which power is lost to joule heating. The *shape factor* G is quoted in handbooks for various practical geometries.

PROBLEM

1. (a) Find the general solution for E_z in the case of TM waves propagating in a guide constructed with walls which are good conductors when the cross section is a quadrant of a circle of radius a. Take $\epsilon = 1$ and $\mu = 1$ inside the guide.

   (b) Calculate the lowest frequency that can be propagated in a TM mode in this guide when the radius of the quadrant is 5 cm.